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ON COMPACT ENGEL GROUPS

BY

YURI MEDVEDEV

EBS, Bank of Montreal, 3300 Bloor St. W., 14th Floor, West Tower
Toronto, Ontario, Canada M8X 2X3
e-mail: yuri.medvedev@bmo.com

ABSTRACT

In 1992, Wilson and Zelmanov proved that a profinite Engel group is locally nilpotent. Here we prove the stronger result that every compact Engel group is locally nilpotent.

1. Introduction

A group is called **Engel** if every two elements a, b of the group satisfy a relation of the form

$$[\dots[[a,b],b],\dots,b]=1,$$

where $[a,b]=a^{-1}b^{-1}ab$, the **commutator** of a and b. Following the "left-normed" convention, we write the left-hand side of the above Engel relation as $[a,b,\dots,b]$ or, more briefly, as $[a,{}_nb]$, where n denotes the number of entries of b. In general the number n may depend on both elements a and b. If for some positive integer n the identical relation $[x,{}_ny]\equiv 1$ holds for all elements x and y of the group, we call the group n**-Engel**, and say that the group satisfies the n**-Engel law**. Thus we have two types of Engel condition. By applying the main result from [1] (corrected in [2]) one obtains a structure result for compact groups satisfying an Engel law. Indeed, since every compact group is a subdirect product of closed subgroups of unitary matrix groups ([3, Corollary 22.14, p. 345]) and by Mal'cev's Theorem (see for instance [7]) a finitely generated linear group is residually finite, we have immediately that compact groups are locally residually finite, and so lie in the "good" class of "locally graded" groups (see [2]). By the

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main result from [1, 2] characterizing the n-Engel groups from that class, we thus immediately obtain the following

PROPOSITION: *A compact n-Engel group is both nilpotent-by-finite exponent and finite exponent-by-nilpotent.*

Our main result concerns compact groups satisfying the more general Engel condition. In [8] Wilson and Zelmanov proved that a profinite Engel group is locally nilpotent. Here we prove the following stronger result.

THEOREM: *Every compact Engel group is locally nilpotent.*

As an immediate consequence we have the

COROLLARY: *A compact group is locally nilpotent if and only if each of its two-generator subgroups is nilpotent.*

2. Preliminary facts

For us all topological groups are assumed Hausdorff. Throughout the paper all neighbourhoods will be assumed open. For basic facts about the structure of topological groups we refer to [3]. If U is any open neighborhood of 1 in a topological group, then it is easy to see that there exists a symmetric neighborhood U_s of 1 contained in U, i.e., such that $U_s = U_s^{-1} \subseteq U$. If the group is compact we can choose a symmetric neighborhood U_s which is also invariant under all inner automorphisms of the group (see [3, Theorem 4.9]).

We shall assume throughout that all neighborhoods of 1 in compact groups are symmetric and invariant under the inner automorphisms.

Let C be any compact group, and $a_1, \ldots, a_k$ any k elements of C, where k is finite. Throughout the paper D will denote the abstract subgroup of C generated by $a_1, \ldots, a_k$, and G the subgroup generated as topological group by $a_1, \ldots, a_k$. Thus G is the closure of D in C, and is therefore a compact topological group under the subgroup topology induced from C. Our aim is to show that D, and therefore G, is nilpotent.

For any neighborhood U of 1 in G and any positive integer n, there exists a neighborhood V of 1 such that $V^n \subseteq U$, where V^n is the set of all products of n elements from V. (This can be seen as follows: By the definition of a topological group, for each pair of elements $a, b \in G$ and for each neighborhood U of ab, there exist neighborhoods X of a and Y of b such that $XY \subseteq U$. Taking $a = b = 1$, we obtain, for each neighborhood of 1, neighborhoods X, Y of 1 such that $XY \subseteq U$. Hence for $V := X \cap Y$, we have $V^2 \subseteq U$, and so on.) We write $U(n) := V$, where $V^n \subseteq U$. Since all neighborhoods of 1 are understood to be symmetric and invariant under inner automorphisms, for any $x \in U(2)$ and $y \in G$ we have $[x, y] = x^{-1} \cdot y^{-1}xy \in U(2)^2 \subseteq U$.

The group G is covered by neighborhoods of the form Ug, $g \in G$. Since G is compact there are finitely many elements $d_1, \dots, d_s$ such that $G = \bigcup_{i=1}^{s} Ud_i$. In fact we can for suitably large s always find such elements $d_1, \dots, d_s$ in D. To see this observe first that there exist elements $c_1, \dots, c_s \in G$ such that $G = \bigcup_{i=1}^{s} U(2)c_i$. Then since the group D is dense in G we can find $d_i \in D \bigcap U(2)c_i$ for $i = 1, \dots, s$, whence $U(2)c_i \subseteq U(2)U(2)^{-1}d_i \subseteq U(2)^2 d_i \subseteq Ud_i$. Hence $G = \bigcup_{i=1}^{s} Ud_i$.

We denote by $\gamma_n(K)$ the nth term of the lower central series of any group K.

For each positive integer n consider the closed subset

$$T_n = \{(x, y) \in G \times G \,|\, [x, {}_n y] = 1\}.$$

Since G is Engel, the group $G \times G$ is the union of countably many closed subsets $T_1, T_2, \dots$. By the Baire Category Theorem (see for example [4]) at least one of these subsets, say T_m, contains an open neighborhood. Hence we can find elements $g_1, g_2 \in D$, and a neighborhood U of 1 in G such that for all elements $x, y \in U$ the following relation holds:

$$(1) \qquad [xg_1, {}_m yg_2] = 1.$$

Throughout the paper we hold fixed the neighborhood U and the elements $g_1, g_2 \in D$.

The following commutator relations, valid in any group for any elements x, y, z, will be used repeatedly:

$$(2) \qquad xy = yx[x, y], \quad [x, yz] = [x, z][x, y][x, y, z], \quad [xy, z] = [x, z][x, z, y][y, z].$$

Our proof will use the notion of the **locally nilpotent radical** of an $\mathbb{N}$-graded Engel Lie ring. Let $L = \sum L_i$ be a $\mathbb{N}$-graded Lie ring, that is, such that $[L_i, L_j] \subseteq L_{i+j}$ for every i and j. We define such a graded Lie ring L to be **Engel** if every homogeneous element $x_i \in L_i$ is Engel, that is, $[y, {}_n x_i] = 0$ for some positive integer $n = n(x_i)$ depending only on x_i, and for every $y \in L$. An ideal I of L is **homogeneous** if it has the form $I = \sum I_i$ where $I_i \subseteq L_i$. The analogue for Engel Lie rings of the following proposition is well-known, and goes over to $\mathbb{N}$-graded Lie rings via essentially the same proof (see [6, 10, 12]).

PROPOSITION 2.1: *Every $\mathbb{N}$-graded Engel Lie ring L has a unique homogeneous locally nilpotent ideal $Loc(L)$ which contains all other homogeneous locally nilpotent ideals. The quotient ring $L/Loc(L)$ contains no nonzero homogeneous locally nilpotent ideals.*

The ideal $Loc(L)$ is called the **locally nilpotent radical** of L.

3. Nilpotence of the associated Lie ring

Let $D_i = \gamma_i(D)$ denote the i-th term of the lower central series of D. Consider the Lie ring L associated with D:

$$L = D_1/D_2 \oplus D_2/D_3 \oplus \cdots.$$

The Lie ring structure on L is defined via the brackets

$$[x_i D_{i+1}, y_j D_{j+1}] := [x_i, y_j] D_{i+j+1},$$

where $x_i \in D_i$ and $y_j \in D_j$. The Lie ring L is thus generated by the finite set $\{a_1 D_2, \ldots, a_k D_2\}$, where $a_1, \ldots, a_k$ are, as above, the generators of D.

In this section we show that the Lie ring L is nilpotent. To this end, we first establish that the graded Lie ring L is Engel, and then deduce that L coincides with its locally nilpotent radical. Since L is finitely generated it will then follow that it is nilpotent.

LEMMA 3.1: *Corresponding to each neighborhood V of 1 in G, there exists a positive integer $t = t(V)$ satisfying the following condition: for each pair of elements $a, x \in G$, there exist integers $0 \leq n_1 < n_2 \leq t$ such that $[x, {}_{n_1} a] = v[x, {}_{n_2} a]$ for some $v \in V$.*

Proof: Since the group G is compact there exist elements $g_1, \ldots, g_t$ such that $G = \bigcup_{i=1}^{t} V(2) g_i$. Therefore, there exist integers $0 \leq n_1 < n_2 \leq t$ such that $[x, {}_{n_1} a] = v_1 g_i$ and $[x, {}_{n_2} a] = v_2 g_i$ for some i and some $v_1, v_2 \in V(2)$. Hence $[x, {}_{n_1} a] = v[x, {}_{n_2} a]$ where $v = v_1 v_2^{-1} \in V(2)^2 \subseteq V$. ∎

LEMMA 3.2: *The $\mathbb{N}$-graded Lie ring L is Engel.*

Proof: Given any element $a \in G$ and any positive integer i, consider the closed set $T_i = \{x \in G | [x, {}_i a] = 1\}$. Since the group G is Engel, it is the union of countably many closed subsets $T_1, T_2, \ldots$. As before, by the Baire Category Theorem one of these subsets, T_n say, contains an open neighborhood. Hence there exists a neighborhood V of 1 in G and an element $g \in G$ such that for every $v \in V$ the following relation holds:

$$[vg, {}_n a] = 1. \tag{3}$$

By an easy induction using the relations (2), a commutator of the form $[xy, {}_n z]$ can be shown to be expressible as

$$[xy, {}_n z] = [x, {}_n z][y, {}_n z]\rho(x, y, z), \tag{4}$$

where ρ is a product of commutators in x, y and z each involving all of x, y and z, with z occurring in each at least n times. Since $1 \in V$, we may take $v = 1$ in (3), obtaining $[g, {}_n a] = 1$. Hence

$$1 = [vg, {}_n a] = [v, {}_n a][g, {}_n a]\rho(v, g, a) = [v, {}_n a]\rho(v, g, a),$$

that is, for each $v \in V$ we have

$$[v, {}_n a] = \rho(v, g, a)^{-1}, \tag{5}$$

where ρ is a product of commutators in v, g and a, each involving all of v, g, a, with a occurring in each at least n times. By Lemma 3.1, corresponding to each element $x \in G$ there exist integers n_1, n_2 with $0 \leq n_1 < n_2 \leq t = t(V)$, such that $[x, {}_{n_1} a] = v[x, {}_{n_2} a]$ for some $v \in V$. Setting $v = [x, {}_{n_1} a][x, {}_{n_2} a]^{-1}$ in (5), we see that the commutator $[[x, {}_{n_1} a][x, {}_{n_2} a]^{-1}, {}_n a]$ can be rewritten as a product of commutators in x, g and a, each involving all of x, g, a, with a occurring in each at least $n_1 + n$ times. On the other hand, by (4)

$$[[x, {}_{n_1} a][x, {}_{n_2} a]^{-1}, {}_n a] = [[x, {}_{n_1} a], {}_n a][[x, {}_{n_2} a]^{-1}, {}_n a]\rho([x, {}_{n_1} a], [x, {}_{n_2} a]^{-1}, a),$$

whence

$$[[x, {}_{n_1} a], {}_n a] = [[x, {}_{n_1} a][x, {}_{n_2} a]^{-1}, {}_n a]\rho([x, {}_{n_1} a], [x, {}_{n_2} a]^{-1}, a)^{-1}[[x, {}_{n_2} a]^{-1}, {}_n a]^{-1}.$$

By the preceding observation concerning the commutator $[[x, {}_{n_1} a][x, {}_{n_2} a]^{-1}, {}_n a]$, and since $n_1 < n_2$, the last equation can be used to express the commutator $[[x, {}_{n_1} a], {}_n a] = [x, {}_{n_1+n} a]$ as a product of commutators in x, g, and a of weight $> n_1+n+1$, each involving x and a, with a occurring in each at least n_1+n times. For every x the integer $n_1 < t = t(V)$. Hence for each $x \in G$ the commutator $[x, {}_{t+n} a]$ can be rewritten as a product of commutators in x, g, and a of weight $> t+n+1$, each involving x and a, with a occurring in each at least $t+n$ times. Taking $a \in D_i$, and writing $\bar{a} := aD_{i+1} \in L_i$, the last statement implies that

$$[x, {}_{t+n} \bar{a}] = 0,$$

for every $x \in L$. ∎

From this lemma and Proposition 2.1 we infer immediately

COROLLARY 3.1: *The graded Lie ring L has a locally nilpotent radical $Loc(L)$.*

Now let U be the neighborhood of 1 in G specified in Section 2, and write $V := U(m)$. Thus for any elements $v_1, \dots, v_m \in V = U(m)$, the product $v_1 \dots v_m$ belongs to U, and by (1), for any element $v_0 \in U$, we have

$$[v_0 g_{1,m} v_1 \cdots v_m g_2] = 1.$$

By means of (1), namely $[x g_{1,m} y g_2] = 1$ for all $x, y \in U$, and the identities (2), the last relation can be translated into the following one:

$$\prod_{\sigma \in Sym(m)} [v_0, v_{\sigma(1)}, \dots, v_{\sigma(m)}] = \delta(v_0, v_1, \dots, v_m, g_1, g_2)\theta, \tag{6}$$

where δ is a product of commutators in $v_0, v_1, \dots, v_m, g_1, g_2$, each involving all of $v_0, v_1, \dots, v_m$ and of weight at least $m+2$, and θ is a product of commutators in $v_0, v_1, \dots, v_m, g_1, g_2$, each of which does not involve at least one of $v_0, v_1, \dots, v_m$. Without loss of generality we can assume that $\theta = \theta_0 \theta_0'$, where θ_0 is a product of such commutators involving v_0, and θ_0' is a product of such commutators not involving v_0. Since $1 \in V$, by putting $v_0 = 1$ in (6) we infer that $\theta_0' = 1$. We may therefore assume that all commutators in θ involve v_0. Repeating this argument for each $v_1, \dots, v_m$ in turn, we see that $\theta = 1$. Hence for any elements $v_0, \dots, v_m \in V$ we have the relation

$$\prod_{\sigma \in Sym(m)} [v_0, v_{\sigma(1)}, \dots, v_{\sigma(m)}] = \delta(v_0, v_1, \dots, v_m, g_1, g_2), \tag{7}$$

where δ is a product of commutators in $v_0, v_1, \dots, v_m, g_1, g_2$, each involving all of $v_0, v_1, \dots, v_m$, and of weight $\geq m+2$.

As observed in Section 2, there exist elements $d_1, \dots, d_t \in D$ such that $G = \bigcup_{i=1}^t V(2)d_i$. (Note that by the proof of Lemma 3.1 we may take this number t to be the number $t(V)$ of that lemma.)

For each element $a \in G$ we choose a fixed element $d(a) \in \{d_1, \dots, d_t\}$ such that $a \in V(2)d(a)$, and define the subset

$$\Delta(a) := \{d([a, x_1, \dots, x_s]) |\, x_1, \dots, x_s \in G,\ s \geq 0\} \subseteq \{d_1, \dots, d_t\}.$$

(Here for $s = 0$, we set $[a, x_1, \dots, x_s] := a$.)

We are now ready to prove the main lemma .

LEMMA 3.3: *The Lie ring L is nilpotent.*

Proof: It suffices to show that the finitely generated, graded, Engel Lie ring L coincides with its locally nilpotent radical $Loc(L)$. This will follow from the fact that for an arbitrary element $a \in \gamma_r(D) = D_r$, the element $\bar{a} := aD_{r+1} \in L_r$ generates a locally nilpotent ideal in the Lie ring L. We establish this fact by induction on the number $|\Delta(a)|$ of elements in the set $\Delta(a)$.

We begin with the initial step $|\Delta(a)| = 1$, that is, $\Delta(a) = \{d\}$ for some $d \in \{d_1, \dots, d_t\}$.

Let I denote the ideal in L generated by the element $\bar{a}$. The ideal I is spanned by elements of the form $\bar{c} = cD_{j+1}$ with $c = [a, a_{i_1}, \dots, a_{i_s}]$, where $a_{i_1}, \dots, a_{i_s} \in D$, $s \geq 0$, and $j = r + s$. For any $m+1$ elements $\bar{c}_0 = c_0 D_{j_0+1}$, $\bar{c}_1 = c_1 D_{j_1+1}, \dots, \bar{c}_m = c_m D_{j_m+1}$ of this form, we have $c_0 = u_{01}d, c_1 = u_{11}d, \dots, c_m = u_{m1}d$ for some $u_{01}, u_{11}, \dots, u_{m1} \in V(2)$. Since $d([c_i, a_1]) = d$ for all i, we have again $[c_0, a_1] = u_{02}d, [c_1, a_1] = u_{12}d, \dots, [c_m, a_1] = u_{m2}d$, for some $u_{02}, u_{12}, \dots, u_{m2} \in V(2)$. Hence $c_i[c_i, a_1]^{-1} = u_{i1}u_{i2}^{-1} \in V$, for $i = 0, 1, \dots, m$. Putting $v_i = c_i[c_i, a_1]^{-1}$ for $i = 0, 1, \dots, m$ in (7), we infer that the product

$$\prod_{\sigma \in Sym(m)} [c_0[c_0, a_1]^{-1}, c_{\sigma(1)}[c_{\sigma(1)}, a_1]^{-1}, \dots, c_{\sigma(m)}[c_{\sigma(m)}, a_1]^{-1}]$$

can be rewritten as a product of commutators in $c_0, c_1, \dots, c_m$, a_1, g_1, g_2, of weight $\geq m+2$, each involving all of $c_0, c_1, \dots, c_m$. On the other hand, using the relations (2) this product can be expressed as a product of

$$\prod_{\sigma \in Sym(m)} [c_0, c_{\sigma(1)}, \dots, c_{\sigma(m)}]$$

with certain other commutators of weight $\geq m+2$, involving all of $c_0, c_1, \dots, c_m$. Hence the product $\prod_{\sigma \in Sym(m)} [c_0, c_{\sigma(1)}, \dots, c_{\sigma(m)}]$ can be rewritten as a product of commutators of weight $\geq m+2$, involving all of $c_0, c_1, \dots, c_m$. In the Lie ring L this gives

$$\sum_{\sigma \in Sym(m)} [\bar{c}_0, \bar{c}_{\sigma(1)}, \dots, \bar{c}_{\sigma(m)}] = 0.$$

Since elements of this form span the ideal I, we infer that the Engel ideal I satisfies the linearized m-Engel condition, and therefore, by a result of Zelmanov [14, Proposition 2], is locally nilpotent. Hence $\bar{a} \in Loc(L)$.

For the inductive step we assume that $|\Delta(a)| = q+1$ for some $q \geq 1$, and that for all j and all $b \in D_j$ such that $|\Delta(b)| \leq q$, the element $\bar{b} = bD_{j+1}$ belongs to $Loc(L)$. Again we consider the ideal I in L generated by the element $\bar{a}$. Write

$K := I \bigcap Loc(L)$. The quotient Lie ring I/K is spanned by elements of the form $\bar{c} + K$, where $\bar{c} := cD_{j+1}$ for some D_{j+1}, and $c = [a, a_{i_1}, \ldots, a_{i_s}]$ for some $a_{i_1}, \ldots, a_{i_s} \in D$. We want to show that in fact $I \subseteq K$. Assuming this is not the case, consider $m+1$ arbitrary nonzero elements of this form, say $\bar{c}_i + K$, where $c_i \in D_{r_i}$ and $\bar{c}_i = c_i D_{r_i+1}$ for $i = 0, 1, \ldots, m$. Since for every i the element $\bar{c}_i \notin K \subseteq Loc(L)$, we must have $|\Delta(c_i)| = q+1$. Since $\Delta(c_i) \subseteq \Delta(a)$ and $\Delta(c_i)$ contains the same number $q+1$ of elements as $\Delta(a)$, we have $\Delta(c_i) = \Delta(a)$ for every i.

For every i, the ideal generated by $\bar{c}_i$ cannot be locally nilpotent, since otherwise, by Proposition 2.1, the element $\bar{c}_i$ would lie in K. Hence also $[\bar{c}_i, L] \nsubseteq K$, since otherwise $\bar{c}_i$ would represent a homogeneous nonzero central element of $L/Loc(L)$, which is ruled out by the final assertion of Proposition 2.1. Since the Lie ring L is generated by the finitely many elements $\bar{a}_1 = a_1 D_2, \ldots, \bar{a}_k = a_k D_2$, there exists for each i an element $a_{i1} \in \{a_1, \ldots, a_k\}$ such that

$$[\bar{c}_i, \bar{a}_{i1}] = [c_i, a_{i1}] D_{r_i+2} \notin K.$$

Hence $|\Delta([c_i, a_{i1}])| = q+1$. Since $\Delta([c_i, a_{i1}]) \subseteq \Delta(a)$, and both sets contain the same number $q+1$ of elements, we deduce that $\Delta([c_i, a_{i1}]) = \Delta(c_i) = \Delta(a)$ for each i. It follows that if $c_i = u_{i1} d(c_i)$ for some $u_{i1} \in V(2)$, then for some $x_{i2}, \ldots, x_{is} \in G$ the element $b_i := [c_i, a_{i1}, x_{i2}, \ldots, x_{is}] = u_{i2} d(c_i)$, for some $u_{i2} \in V(2)$. Hence $c_i b_i^{-1} = u_{i1} u_{i2}^{-1} \in V$ for all i. Making the substitution $v_i = c_i b_i^{-1}$ for $i = 0, 1, \ldots, m$, in (7), we infer that the product

$$\prod_{\sigma \in Sym(m)} [c_0 b_0^{-1}, c_{\sigma(1)} b_{\sigma(1)}^{-1}, \ldots, c_{\sigma(m)} b_{\sigma(m)}^{-1}]$$

can be rewritten as a product of commutators of weight $\geq m+2$, each involving all of $c_0, c_1, \ldots, c_m$. On the other hand, using the relations (2) this product can be expressed as a product of

$$\prod_{\sigma \in Sym(m)} [c_0, c_{\sigma(1)}, \ldots, c_{\sigma(m)}]$$

and certain commutators of weight $\geq m+2$, each involving all of $c_0, c_1, \ldots, c_m$. Hence the product $\prod_{\sigma \in Sym(m)} [c_0, c_{\sigma(1)}, \ldots, c_{\sigma(m)}]$ can be rewritten as a product of commutators of weight $\geq m+2$, each involving all of $c_0, c_1, \ldots, c_m$. In the Lie ring L this yields

$$\sum_{\sigma \in Sym(m)} [\bar{c}_0, \bar{c}_{\sigma(1)}, \ldots, \bar{c}_{\sigma(m)}] = 0.$$

Since the latter equation implies that the quotient Lie ring I/K satisfies the linearized m-Engel condition, we infer as before from Zelmanov [14, Proposition 2] that the Engel ring I/K is locally nilpotent. Since the ideal K is locally nilpotent and the graded ideal I is Engel, we deduce that the ideal I is itself locally nilpotent, whence $I \subseteq Loc(L)$. ∎

4. Proof of the Theorem

The group G is nilpotent if and only if the dense finitely generated subgroup D is nilpotent. By Lemma 3.3 there exists a positive integer c such that $\gamma_c(D) \subseteq \gamma_i(D)$ for every i. It thus suffices to observe that the group D is residually nilpotent. This follows from the fact noted in the introduction, that G is a subdirect product of groups of unitary matrices, so that its subgroup D is a subdirect product of finitely generated linear groups, and hence residually finite. Since D is also Engel, and finite Engel groups are nilpotent (Zorn's theorem), it follows that D is residually nilpotent, and therefore nilpotent of class at most c. ∎

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References

[1] R. G. Burns and Yu. Medvedev, *A note on Engel groups and local nilpotence*, Journal of the Australian Mathematical Society (Series A) **64** (1998), 92–100.

[2] R. G. Burns and Yu. Medvedev, *Group laws implying virtual nilpotence*, Journal of the Australian Mathematical Society (Series A), (2003) to appear.

[3] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol. I, Springer-Verlag, New York, Berlin, 1963.

[4] J. L. Kelley, *General Topology*, Van Nostrand, New York, 1955.

[5] A. I. Kostrikin, *On Lie rings with Engel's condition*, Doklady Akademii Nauk SSSR **108** (1956), no.4.

[6] B. I. Plotkin, *Algebraic sets of elements in groups and Lie algebras*, Uspekhi Matematicheskikh Nauk **13** (1958), no.6.

[7] B. A. F. Wehrfritz, *Infinite Linear Groups*, Springer-Verlag, New York, Berlin, 1973.

[8] J. S. Wilson and E. I. Zelmanov, *Identities for Lie algebras of pro-p groups*, Journal of Pure and Applied Algebra **81** (1992), 103–109.

[9] E. I. Zelmanov, *Engel Lie algebras*, Doklady Akademii Nauk SSSR **292** (1987), 265–268.

[10] E. I. Zelmanov, *On some problems of group theory and Lie algebras*, Mathematics of the USSR-Sbornik **66** (1990), 159–168.

[11] E. I. Zelmanov, *The solution of the restricted Burnside problem for groups of odd exponent*, Mathematics of the USSR-Izvestiya **36** (1991), 41–60.

[12] E. I. Zelmanov, *The solution of the restricted Burnside problem for 2-groups*, Matematicheskii Sbornik **182** (1991), 568–592.

[13] E. I. Zelmanov, *Nil Rings and Periodic Groups*, KMS Lecture Notes in Mathematics, Korean Mathematical Society, Seoul, 1992.

[14] E. I. Zelmanov, *On periodic compact groups*, Israel Journal of Mathematics **77** (1992), 83–95.